The Logic of Bunched Implications is Undecidable

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LICS

Plan for the talk

- Bunched Implication logic (BI): What is it and why is it(s decision problem) interesting?
- Broader setting
- Results and technique

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Language (additives, multiplicatives)

$$p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid 1 \mid \varphi * \varphi \mid \varphi \mathrel{-\!\!*} \varphi$$

Algebraic semantics for BI

Algebraic semantics: A BI-algebra is an algebra

$$\mathbf{A} = (A, \wedge, \vee, \rightarrow, \top, \bot, *, -\!*, 1)$$

where:

- $(A, \wedge, \vee, \rightarrow, \top, \bot)$ is a Heyting algebra
- $(A, *, 1)$ is a commutative monoid
- $*$ is residuated by $-\!*$:

$$x * y \leqslant z \qquad \text{iff} \qquad y \leqslant x -\!* z.$$

Fact: $\vdash \varphi \Rightarrow \psi$ iff BI $\models \varphi \leqslant \psi$.

We use: $\nvdash \top \Rightarrow \psi$ iff BI $\not\models \psi$.

Goal: Prove (un)decidability of BI-provability.
We prove undecidability for BI $\not\models \psi$.

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Broader scope

Broader scope, broader theme: Understanding the decidability/undecidability boundary.

Upper bound

Recall: $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \top, \bot, *, -\!\!*, 1)$

The BI-algebra $\mathcal{P}_\omega(\mathbb{N})^+$

Let $\mathcal{P}_\omega(\mathbb{N}) := \{X \subseteq \mathbb{N} \mid X \text{ is finite}\}$ and

$$\mathcal{P}_\omega(\mathbb{N})^+ := \big(\mathcal{P}(\mathcal{P}_\omega(\mathbb{N})), \cap, \cup, \rightarrow, \mathcal{P}_\omega(\mathbb{N}), \varnothing, *, -\!\!*, \{\varnothing\}\big),$$

where

- $\rightarrow$ is Boolean implication: $X \rightarrow Y := X^c \cup Y$
- $*$ is point-wise union: $X * Y := \{x \cup y \mid x \in X, y \in Y\}$
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Observe that $\mathcal{P}_\omega(\mathbb{N})^+$ is a BBI-algebra.

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Lower bound and algebraic setting

GBI: $*$ is not assumed commutative, so $-\!*$ splits into two divisions $\backslash, /$.

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- $\cdot$ is residuated by $\backslash, /$: $\quad x \cdot y \leqslant z \quad$ iff $\quad y \leqslant x \backslash z \quad$ iff $\quad x \leqslant z/y$.

A weaker setting suffices:

A disjointive distributive residuated lattice is an algebra

$$(A, \wedge, \vee, \top, \bot, \neg, \cdot, \backslash, /)$$

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How will we prove undecidability?

The Tiling Problem

- A Wang tile is a square with colors on each side.
- The tiling problem: Given a finite set of Wang tiles $\mathcal{W}$, is it possible to tile the first quadrant $\mathbb{N} \times \mathbb{N}$ so that adjacent tiles match along their shared edges
- Introduced by Wang (1961, 1963) and proven undecidable by Berger (1966).

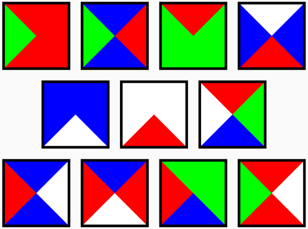

Figure 1: Wang tiles

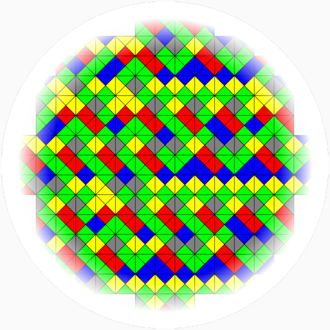

Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

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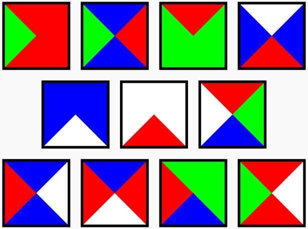

Figure 1: Wang tiles

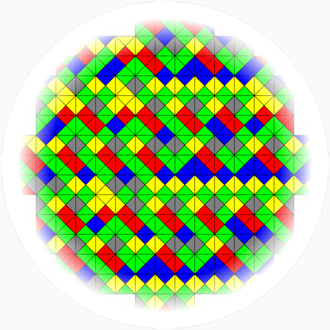

Figure 2: A tiling of the plane

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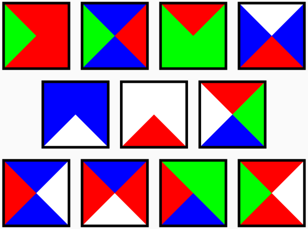

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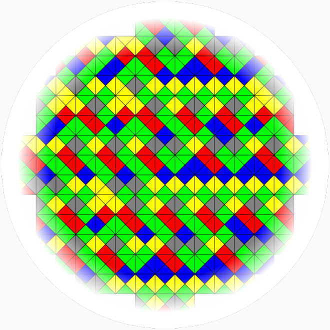

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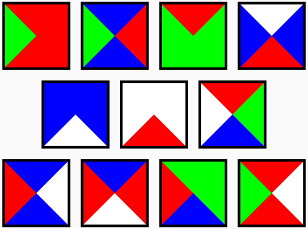

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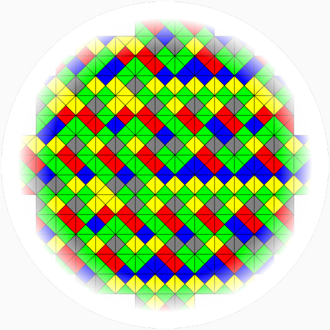

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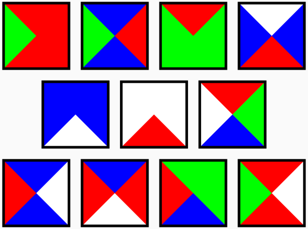

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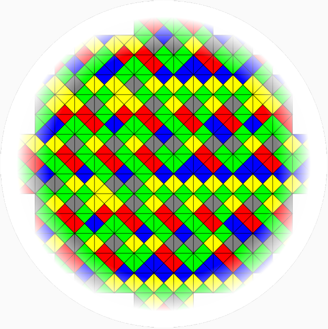

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Given $\mathcal{W}$, construct a formula $\phi_{\mathcal{W}}$ such that:

$$\begin{tikzcd}[column sep=small]
& \boxed{\begin{array}{c}\mathbf{A} \not\models \phi_{\mathcal{W}} \text{ for } \mathbf{A} \text{ a disjointive} \\ \text{distr. residuated lattice}\end{array}} \arrow[dr, Rightarrow] & \\
\boxed{\mathcal{P}_\omega(\mathbb{N})^+ \not\models \phi_{\mathcal{W}}} \arrow[ur, Rightarrow] & & \boxed{\begin{array}{c}\mathcal{W} \text{ tiles} \\ \mathbb{N} \times \mathbb{N}\end{array}} \arrow[ll, Leftarrow]
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Theorem

Every class of disjointive distributive residuated lattices that contains $\mathcal{P}_\omega(\mathbb{N})^+$ has an undecidable equational theory.

Main theorem

Given $\mathcal{W}$, construct a formula $\phi_{\mathcal{W}}$ such that:

$\mathbf{A} \not\models \phi_{\mathcal{W}}$ for $\mathbf{A}$ a disjointive distr. residuated lattice

$$\mathcal{P}_\omega(\mathbb{N})^+ \not\models \phi_{\mathcal{W}} \quad \Longleftarrow \quad \mathcal{W} \text{ tiles } \mathbb{N} \times \mathbb{N}$$

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Recall the above. From this, we get:

Theorem

BI is undecidable.

Proof. $\mathcal{P}_\omega(\mathbb{N})^+$ is a BI-algebra, and (reducts of) BI-algebras are, in particular, disjointive distributive residuated lattices.

[In fact, the tiling formulas $\phi_{\mathcal{W}}$ only use $\{\wedge, \vee, \neg, \backslash, /\}$ (or in the commutative setting: $\{\wedge, \vee, \neg, -\!\!*\}$)]

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Theorem

The variety of disjointive distributive residuated lattices has an **undecidable** equational theory.

That is, the mere presence of an operation $\neg$ with $x \wedge \neg x = \bot$ is enough to cross from the decidable to the undecidable.

Duals of disjointive distributive residuated lattices

Relational semantics given by (disjointive associative) frames $\mathfrak{F} = (S, \circ, N)$:

- $\circ : S^2 \to \mathcal{P}(S)$ is associative: $(x \circ y) \circ z = x \circ (y \circ z)$
- $N : \mathcal{P}(S) \to \mathcal{P}(S)$ is disjointive: $X \cap N(X) = \varnothing$.

($\circ$ is lifted to $Y, Z \subseteq S$ by $Y \circ Z := \{x \in S \mid \exists y \in Y, z \in Z : x \in y \circ z\}$)

Given $(\mathfrak{F}, V)$, set:

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Given $(\mathfrak{F}, V)$, set:

$$\begin{aligned} \|p\| &:= V(p) & \|\neg\varphi\| &:= N(\|\varphi\|) \\ \|\varphi \wedge \psi\| &:= \|\varphi\| \cap \|\psi\| & \|\varphi \backslash \psi\| &:= \{s \in S \mid \|\varphi\| \circ s \subseteq \|\psi\|\} \\ \|\varphi \vee \psi\| &:= \|\varphi\| \cup \|\psi\| & \|\psi / \varphi\| &:= \{s \in S \mid s \circ \|\varphi\| \subseteq \|\psi\|\}. \end{aligned}$$

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$s \Vdash p$	iff	$s \in V(p)$
$s \Vdash \neg\varphi$	iff	$s \in N(\lVert\varphi\rVert)$
$s \Vdash \varphi \wedge \psi$	iff	$s \Vdash \varphi$ and $s \Vdash \psi$
$s \Vdash \varphi \vee \psi$	iff	$s \Vdash \varphi$ or $s \Vdash \psi$
$s \Vdash \varphi\backslash\psi$	iff	$\forall x, z \in S$: if $x \Vdash \varphi$ and $z \in x \circ s$, then $z \Vdash \psi$
$s \Vdash \psi/\varphi$	iff	$\forall y, z \in S$: if $y \Vdash \varphi$ and $z \in s \circ y$, then $z \Vdash \psi$.

Facts:

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$$
\begin{tikzcd}
 & & & & \iddots \\
 & & \langle 2,2\rangle \arrow[r, dash, "x_3"'] \arrow[d, dash, "y_2"] & \langle 3,2\rangle & \\
 & \langle 1,1\rangle \arrow[r, dash, "x_2"'] \arrow[d, dash, "y_1"] & \langle 2,1\rangle & & \\
s \arrow[r, dash, "x_1"'] & \langle 1,0\rangle & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
p \arrow[r, dash, "x_i"'] & q
\end{tikzcd}
\text{ means } q \in x_i \circ p
$$

$$
\begin{tikzcd}
r \arrow[d, dash, "y_i"] \\
q
\end{tikzcd}
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Figure 3: Staircase of grid points.

Lemma 2: $\mathbb{N}^2$ construction

Grid points through associativity:

$$
\begin{tikzcd}
\langle m, n+1 \rangle \arrow[r, dash, dashed, "x_{m+1}"'] & \langle m{+}1, n{+}1 \rangle \\
\langle m, n \rangle \arrow[u, dash, dashed, "y_{n+1}"'] \arrow[r, dash, "x_{m+1}"'] & \langle m{+}1, n \rangle \arrow[u, dash, "y_{n+1}"']
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Method

- Reduction from Wang Tiling
- Encoded in relational semantics

Scope clarified algebraically

Highlights

- BI is undecidable
- GBI is undecidable
- BBI is undecidable (old result, new proof)
- The variety of (disjointive) distributive residuated lattices is (un)decidable.

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Thank you!